

## Stability of vacua in New Massive Gravity in different gauges

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### Abstract

We consider  $AdS_3$  and warped  $AdS_3$  vacua in new massive gravity and study the highest weight modes and general propagating modes as a set of solutions for the linearized equations of motion. We observed that depending on the choice of gauge there are two types of solution. We show that for warped  $AdS_3$  vacuum, the massless modes which only appear in harmonic gauge have zero energy density and they do not get higher curvature corrections. By computing the energy density it can be shown that all massive modes have negative energy density. Our computations prove that the massless modes for squashed space-time are always stable, while for the stretched space-time there are restrictions on the parameters of the theory to have a stable mode.

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# 1 Introduction

According to *AdS/CFT* correspondence [1] the boundary fields parametrize boundary conditions of the bulk fields and couple to the operators in the dual conformal field theory. The sub-leading radial behavior of these fields at boundary is obtained by finding the most general asymptotic solutions to the field equations. For theories that admit asymptotically locally *AdS* solutions these general solutions, which is sometimes called the Fefferman-Graham expansion, can always be found by solving algebraic equations [2]. Even though higher derivative terms are treated as perturbative corrections to two derivative actions, they do not change the usual *AdS/CFT* setup [3]. In fact the field equation implies that there are some additional boundary data to be specified.

Since the three-dimensional gravity described with Einstein-Hilbert (EH) action has no degrees of freedom [4,5], commonly other deformations of pure EH gravity have been considered. In fact the addition of higher order derivative terms provides the theory with propagating degrees of freedom, i.e. three-dimensional gravitons. The first studied theory was the topologically massive gravity (TMG) which constructed by adding a cosmological constant and a gravitational Chern-Simons term [6]-[8]. The other theory which we are going to consider here, is the New Massive Gravity (NMG) [9]. Also some extended theories of NMG have been discussed in [10], [11], [12]. The quantization of such theories seems to give a richer structure than the EH theory and provides interesting toy models for higher-dimensional theories of quantum gravity.

The *AdS*<sub>3</sub> vacuum is the first known solution of three-dimensional massive gravity models. Another vacuum solution of the higher order derivative actions is the warped *AdS*<sub>3</sub>. These vacua admit black hole solutions known as BTZ [8] and warped *AdS*<sub>3</sub> black holes [13, 14]. In this paper we are going to discuss about the asymptotic behavior of the metric fluctuations in NMG model. Since the corresponding black holes have equivalent asymptotic behavior as vacuum solutions we consider only the vacua here.

To find the behavior of metric fluctuations around a background there are two approaches. In the first approach one finds the highest weight modes which are belong to some representations of the isometry group of the background. According to the isometry group of each background ( $SL(2, R)_L \times SL(2, R)_R$  for *AdS*<sub>3</sub> and  $SL(2, R) \times U(1)$  for warped *AdS*<sub>3</sub>) there are some generators. The highest weight modes are those which are killed by raising ladder operators ( $L_1$  and  $\bar{L}_1$  for *AdS*<sub>3</sub> and  $L_1$  for warped). Additionally these modes must be the solutions of the linearized equations of motion. Of course there is no reason that all solutions to the equations of motion fall in the highest weight representation.

In the other approach, one substitutes the metric fluctuations as eigen-modes of energy and momentum in the linearized equations of motion and tries to find a decoupled differential equation for each component of the perturbations. The regular solution at the asymptotic limit can be obtain by using the Frobenius's method. In the asymptotic limit where the radial direction,  $r$ , approaches to the boundaries, solutions behave like  $r^{-B}$ , where the parameter  $B$  is a function of constants in the theory and it can be shown that it is closely related to the frequencies in the highest weight approach. These asymptotic solutions often called general propagating

modes. In the case of TMG the calculations for warped  $AdS_3$  using these two approaches has been discussed in [15].

The stability around a certain background depends on the selection of consistent boundary conditions. For example there exist several consistent choices of boundary conditions for three dimensional massive gravity models in  $AdS_3$  background. But only more confining ones exclude negative energy modes (unstable modes), which are different to Brown-Henneaux (BH) boundary conditions [16]. Similarly the massive propagating modes of warped  $AdS_3$  do not obey the Compère-Detournay (CD) boundary conditions [17,18]. In fact the BH and CD boundary conditions are only consistent with the pure large gauge in which the asymptotic perturbations are given by the Lie derivative of the background metric  $h_{\mu\nu} = \mathcal{L}_\xi \bar{g}_{\mu\nu}$  and  $\xi^\mu$  is the non-vanishing asymptotic diffeomorphism. We have been considered these boundary conditions in [11] which lead to the central charges of the dual CFT living at the boundary of asymptotically  $AdS_3$  space-times. Note that for the propagating modes the metric perturbations cannot be written in this form for any  $\xi^\mu$  of background metric  $\bar{g}_{\mu\nu}$ .

The stability of TMG around  $AdS_3$  vacuum in harmonic gauge has been shown in [19]. They have obtained a stable theory without negative energy at chiral point  $\mu l = 1$ . Calculations for stability of BTZ black hole in NMG has been done in [21], [22].

In this paper we will study the perturbations around  $AdS_3$  and warped- $AdS_3$  backgrounds in NMG for two different gauge conditions, the harmonic (transverse) gauge given by  $\nabla_\mu h^{\mu\nu} = 0$  and the other gauge which is given by  $h_{\mu\varphi} = 0$  and check the stability of different modes.

This paper is organized as follows: In section 2 we briefly discuss the Lagrangian of NMG and its equations of motion and extract the linearized form of these equations around an arbitrary background. In section 3 we review and study  $AdS_3$  vacuum and its perturbations in two mentioned gauges, then check the stability of this vacuum. In section 4 we repeat all steps in section three but for warped- $AdS_3$  vacuum. In section 5 we discuss the stability conditions for warped solution in different gauges. We compute the energy density for the corresponding fluctuations. In Section 6 we will consider extended NMG model which contains sixth order of derivative terms and study its effects on different modes. section 7 includes summary and discussions.

## 2 NMG and its linearized equations of motion

The new massive gravity is described by the following Lagrangian [9]

$$\mathcal{L} = \sqrt{-g} \left( R - 2\Lambda + \kappa_1 R^2 + \kappa_2 R_{\mu\nu} R^{\mu\nu} \right), \quad (2.1)$$

where  $\Lambda$  is the cosmological constant. In addition to a three dimensional gravitational coupling constant, there are two other coupling constants,  $\kappa_1 = \frac{3}{8m^2}$  and  $\kappa_2 = -\frac{1}{m^2}$  which are specified by a mass parameter  $m$ . The equations of motion corresponding to variation of gravitational field are

$$\begin{aligned} T_{\mu\nu}^{NMG} &= R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} + 2\kappa_1 R(R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R) + (2\kappa_1 + \kappa_2)(g_{\mu\nu}\Box - \nabla_\mu \nabla_\nu)R \\ &+ \kappa_2 \Box(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R) + 2\kappa_2(R_{\mu\rho\nu\sigma} - \frac{1}{4}g_{\mu\nu}R_{\rho\sigma})R^{\rho\sigma}. \end{aligned} \quad (2.2)$$

Since our goal in this paper is the study of behavior in the gravitational fluctuations, we need to linearize the equations of motion around an arbitrary background. Let  $\bar{g}_{\mu\nu}$  be a background metric and its fluctuation is described through  $g_{\mu\nu} \equiv \bar{g}_{\mu\nu} + h_{\mu\nu}$ . The linearized Christoffel connection, Riemann and Ricci tensors are

$$\begin{aligned}\delta\Gamma_{\rho\sigma}^{\mu} &= \frac{1}{2}(\bar{\nabla}_{\rho}h_{\sigma}^{\mu} + \bar{\nabla}_{\sigma}h_{\rho}^{\mu} - \bar{\nabla}^{\mu}h_{\rho\sigma}), \quad \delta R^{\mu}_{\rho\nu\sigma} = \bar{\nabla}_{\nu}\delta\Gamma_{\rho\sigma}^{\mu} - \bar{\nabla}_{\sigma}\delta\Gamma_{\rho\nu}^{\mu}, \\ \delta R_{\mu\nu} &= \frac{1}{2}(\bar{\nabla}_{\alpha}\bar{\nabla}_{\mu}h_{\nu}^{\alpha} + \bar{\nabla}_{\alpha}\bar{\nabla}_{\nu}h_{\mu}^{\alpha} - \bar{\nabla}_{\mu}\bar{\nabla}_{\nu}h - \bar{\nabla}_{\alpha}\bar{\nabla}^{\alpha}h_{\mu\nu}), \quad \delta R = \bar{g}^{\alpha\beta}\delta R_{\alpha\beta} - h^{\alpha\beta}\bar{R}_{\alpha\beta}.\end{aligned}\quad (2.3)$$

All derivatives are taken with respect to the background metric  $\bar{g}_{\mu\nu}$ . Using these relations the linearized equations of motion for NMG Lagrangian become

$$\begin{aligned}\delta T_{\mu\nu}^{NMG} &= \delta R_{\mu\nu} - \frac{1}{2}h_{\mu\nu}(\bar{R} - 2\Lambda) - \frac{1}{2}\bar{g}_{\mu\nu}\delta R + 2\kappa_1(\bar{R}_{\mu\nu}\delta R + \bar{R}\delta R_{\mu\nu} - \frac{1}{4}h_{\mu\nu}\bar{R}^2 - \frac{1}{2}\bar{g}_{\mu\nu}\bar{R}\delta R) \\ &+ (2\kappa_1 + \kappa_2)(\bar{g}_{\mu\nu}\bar{\square} - \bar{\nabla}_{\mu}\bar{\nabla}_{\nu})\delta R + \kappa_2(\bar{\square}\delta R_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{\square}\delta R - h^{\alpha\beta}\bar{\nabla}_{\alpha}\bar{\nabla}_{\beta}\bar{R}_{\mu\nu} - \bar{g}^{\alpha\beta}\delta\Gamma_{\alpha\beta}^{\lambda}\bar{\nabla}_{\lambda}\bar{R}_{\mu\nu} \\ &- \delta\Gamma_{\alpha\mu}^{\lambda}\bar{\nabla}^{\alpha}\bar{R}_{\lambda\nu} - \delta\Gamma_{\alpha\nu}^{\lambda}\bar{\nabla}^{\alpha}\bar{R}_{\mu\lambda} - \bar{\nabla}^{\alpha}(\delta\Gamma_{\beta\mu}^{\lambda}\bar{R}_{\lambda\nu} + \delta\Gamma_{\beta\nu}^{\lambda}\bar{R}_{\mu\lambda})) + 2\kappa_2(\delta R_{\mu\rho\nu\sigma}\bar{R}^{\rho\sigma} + \bar{R}_{\mu\nu}^{\rho\sigma}\delta R_{\rho\sigma} \\ &- 2h^{\rho\lambda}\bar{R}_{\mu\rho\nu\sigma}\bar{R}_{\lambda}^{\sigma} - \frac{1}{4}h_{\mu\nu}\bar{R}_{\rho\sigma}^2 + \frac{1}{2}\bar{g}_{\mu\nu}h^{\rho\lambda}\bar{R}_{\lambda}^{\sigma}\bar{R}_{\rho\sigma} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{R}^{\rho\sigma}\delta R_{\rho\sigma}).\end{aligned}\quad (2.4)$$

Everywhere we use the bar notation, it means that a quantity must be computed by background metric.

### 3 $AdS_3$ vacuum in NMG

The global  $AdS_3$  metric is a vacuum solution in new massive gravity

$$ds^2 = l^2 \left[ -(1+r^2)d\tau^2 + \frac{dr^2}{1+r^2} + r^2 d\varphi^2 \right], \quad (3.1)$$

where  $l$  is the radius of  $AdS_3$  space. The isometry group of this space is  $SL(2, R)_L \times SL(2, R)_R$  with the following left and right moving sets of generators

$$\begin{aligned}L_0 &= \frac{i}{2}(\partial_t + \partial_{\varphi}), \quad L_{\pm 1} = \frac{i}{2}e^{\pm i(t+\varphi)} \left[ \frac{r}{\sqrt{1+r^2}}\partial_t \mp i\sqrt{1+r^2}\partial_r + \frac{\sqrt{1+r^2}}{r}\partial_{\varphi} \right], \\ \bar{L}_0 &= \frac{i}{2}(\partial_t - \partial_{\varphi}), \quad \bar{L}_{\pm 1} = \frac{i}{2}e^{\pm i(t-\varphi)} \left[ \frac{r}{\sqrt{1+r^2}}\partial_t \mp i\sqrt{1+r^2}\partial_r - \frac{\sqrt{1+r^2}}{r}\partial_{\varphi} \right].\end{aligned}\quad (3.2)$$

Each set creates a  $SL(2, R)$  Algebra,  $[L_{+1}, L_{-1}] = 2L_0$ ,  $[L_0, L_{\pm 1}] = \mp L_{\pm 1}$ .

Inserting  $AdS_3$  metric (3.1) as a background into the NMG equations of motion (2.2), fixes the value of cosmological constant in terms of the mass parameter and radius of  $AdS_3$  space

$$\Lambda = -\frac{4m^2l^2 + 1}{4m^2l^4}. \quad (3.3)$$

In order to study the linearized equations of motion and find a solution to these equations in a specific background, we need to fix the gauge freedoms. In what follows we will consider two different gauge fixing conditions and compare their results.

### 3.1 Harmonic gauge

The  $AdS_3$  metric due to its symmetries makes the linearized equations of motion (2.4) more simpler. For this metric the curvature tensors are

$$R_{\mu\rho\nu\sigma} = -\frac{1}{l^2}(g_{\mu\nu}g_{\rho\sigma} - g_{\mu\sigma}g_{\rho\nu}), \quad R_{\mu\nu} = -\frac{2}{l^2}g_{\mu\nu}, \quad R = -\frac{6}{l^2}. \quad (3.4)$$

If we compute the trace of linearized equations of motion by multiplying equation (2.4) by  $\bar{g}^{\mu\nu}$  and impose the harmonic gauge condition  $\nabla^\mu h_{\mu\nu} = 0$  then we will find a traceless condition for fluctuations,  $h = h^\mu{}_\mu = 0$ . The harmonic gauge and traceless condition simplify linearized equations of motion (2.4) to a fourth order differential equation as follow [19]

$$(D^{(L)}D^{(R)}D^{(M)}D^{(\tilde{M})}h)_{\mu\nu} = 0. \quad (3.5)$$

The covariant derivative  $D$  defines as a first order operator

$$(D^{(L/R)})_\mu{}^\beta = (\delta_\mu{}^\beta \pm l\varepsilon_\mu{}^{\alpha\beta}\nabla_\alpha), \quad (D^{(M/\tilde{M})})_\mu{}^\beta = (\delta_\mu{}^\beta \pm \frac{1}{\tilde{\mu}}\varepsilon_\mu{}^{\alpha\beta}\nabla_\alpha), \quad (3.6)$$

where  $\varepsilon^{\alpha\beta\gamma} = \frac{1}{\sqrt{g}}\epsilon^{\alpha\beta\gamma}$  with  $\epsilon^{tr\varphi} = 1$ . For NMG Lagrangian one finds  $\tilde{\mu}$  as [20], [21]

$$\tilde{\mu} = \frac{\sqrt{2+4m^2l^2}}{2l}. \quad (3.7)$$

The above differential equation describes one left and one right moving massless graviton mode in  $AdS_3$ . There are also two massive degrees of freedom. These can be seen by the following equations

$$\begin{aligned} (D^{(L)}D^{(R)}h)_{\mu\nu} &= l^2(\nabla^2 - \frac{2}{l^2})h_{\mu\nu} = 0, \\ (D^{(M)}D^{(\tilde{M})}h)_{\mu\nu} &= \frac{1}{\tilde{\mu}^2}(\nabla^2 - \frac{2}{l^2} + \mathcal{M}^2)h_{\mu\nu} = 0, \quad \mathcal{M}^2 = \tilde{\mu}^2 - \frac{1}{l^2} = \frac{4m^2l^2 - 2}{4l^2}. \end{aligned} \quad (3.8)$$

The mass square relation shows that in order to have stability (Tachyon free condition), the parameters of the theory must be bounded to  $m^2l^2 \geq \frac{1}{2}$  values [21].

#### 3.1.1 Highest weight solutions

It is possible to find solutions for linearized equations of motion by requesting the highest weight conditions. In other words these solutions belong to the representations of isometry group of  $AdS_3$  background. The highest weight solutions are eigen-modes of  $SL(2, R)_L \times SL(2, R)_R$  generators with the following eigen-values

$$L_{+1}h_{\mu\nu} = 0, \quad L_0h_{\mu\nu} = \frac{\omega - k}{2}h_{\mu\nu}, \quad \bar{L}_{+1}h_{\mu\nu} = 0, \quad \bar{L}_0h_{\mu\nu} = \frac{\omega + k}{2}h_{\mu\nu}. \quad (3.9)$$

This leads us to consider the following ansatz for metric fluctuations

$$h_{\mu\nu} = e^{-i(\omega t - k\varphi)}g(r) \begin{pmatrix} -(1+r^2)f_1(r) & g_1(r) & g_2(r) \\ g_1(r) & (1+r^2)^{-1}f_2(r) & g_3(r) \\ g_2(r) & g_3(r) & r^2f_3(r) \end{pmatrix}. \quad (3.10)$$

The conditions  $L_{+1}h_{\mu\nu} = \bar{L}_{+1}h_{\mu\nu} = 0$  hold when  $k = 0$ . By solving differential equations from the highest weight conditions, the following behaviors for unknown functions in the ansatz can be achieved

$$\begin{aligned} f_1(r) &= \frac{i}{1+r^2}(C_4 + C_3r^2 + C_5r^4) + \frac{1}{2}r^2C_1 + C_2, \\ f_2(r) &= \frac{1}{2r^2(1+r^2)}((C_1 - 4C_2 - 4iC_3 + 2iC_5)r^4 + (3C_1 - 10C_2 - 6iC_3)r^2 + 2C_1 - 6C_2 - 4iC_3 + 2iC_4), \\ g_1(r) &= \frac{1}{r(1+r^2)}(C_4 + C_3r^2 + C_5r^4), \quad g_2(r) = -iC_5r^4 - \frac{1}{2}r^2(1+r^2)C_1 + iC_4, \\ g_3(r) &= \frac{1}{r(1+r^2)}(C_5r^4 + (iC_1 - 3iC_2 + 2C_3)r^2 + iC_1 - 3iC_2 - C_4 + 2C_3), \end{aligned} \quad (3.11)$$

where we have used the traceless condition to fix  $f_3(r) = -f_1(r) - f_2(r)$ . In this solution,  $C_i$ 's ( $i = 1, \dots, 5$ ) are constants of integrations and function  $g(r)$  must be

$$g(r) = (1+r^2)^{-\frac{\omega}{2}}r^k. \quad (3.12)$$

The harmonic gauge  $\nabla^\mu h_{\mu\nu} = 0$  fixes three of  $C_i$ 's, as

$$C_2 = \frac{i(k+1)(k-2\omega+4)}{2(k-\omega+2)(k-\omega+3)}C_1, C_4 = \frac{i(k^2+(3-\omega)k-\omega^2+2\omega)}{2(k-\omega+2)(k-\omega+3)}C_1, C_5 = \frac{i(k+\omega)}{4(k+2)}C_1. \quad (3.13)$$

Gathering all these results in the linearized equations of motion gives six equations for two unknown constants  $C_1$  and  $C_3$ . One can easily verify that for every arbitrary subset of equations there is a relation of the form

$$\mathcal{M}_{2 \times 2} \begin{pmatrix} C_1 \\ C_3 \end{pmatrix} = 0. \quad (3.14)$$

To have a non-trivial solution ( $C_1 \neq 0, C_3 \neq 0$ ) the determinant of  $\mathcal{M}$  must be zero which gives a relation between the frequencies  $\omega$  and wave numbers  $k$

$$\omega = k, k-2, k+4, k-1 \pm \frac{1}{2}\sqrt{2+4m^2l^2}, k+3 \pm \frac{1}{2}\sqrt{2+4m^2l^2}. \quad (3.15)$$

But there are some constraints which must be considered here. First of all as we mentioned earlier, the highest weight condition holds for  $k = 0$ . On the other hand we expect that the asymptotic metric fluctuations falls off faster than the background metric. We also demand the Tachyon free condition,  $m^2l^2 \geq \frac{1}{2}$ .

Therefor if  $C_1 \neq 0$  then  $\omega > 4$  and we have two possible solutions: Either  $\omega = -1 + \frac{1}{2}\sqrt{2+4m^2l^2}$  for  $m^2l^2 > \frac{49}{2}$ , or  $\omega = 3 + \frac{1}{2}\sqrt{2+4m^2l^2}$  for  $m^2l^2 > \frac{1}{2}$ .

If  $C_1 = 0$  then all constants except  $C_3$  are zero and we need  $\omega > 0$ . Here we have four possible solutions:  $\omega = 4, 3 + \frac{1}{2}\sqrt{2+4m^2l^2}$  and  $\omega = -1 + \frac{1}{2}\sqrt{2+4m^2l^2}$  for  $m^2l^2 > \frac{1}{2}$  and  $\omega = 3 - \frac{1}{2}\sqrt{2+4m^2l^2}$  for  $\frac{17}{2} > m^2l^2 > \frac{1}{2}$ .

### 3.1.2 Asymptotic behavior of propagating solutions

To find the asymptotic behavior of metric perturbations let us to consider the following ansatz

$$h_{\mu\nu} = l^2 e^{-i(\omega t - k\varphi)} \begin{pmatrix} (1+r^2)f_1(r) & g_1(r) & g_2(r) \\ g_1(r) & -(1+r^2)^{-1}f_2(r) & g_3(r) \\ g_2(r) & g_3(r) & -r^2f_3(r) \end{pmatrix}. \quad (3.16)$$

Since we need to know the asymptotic behavior if we drop or keep 1 in  $1 + r^2$  the final results as we have checked will be exactly equal. So for simplicity of calculations one may drop 1. In harmonic gauge, we have three relations among the unknown functions in  $h_{\mu\nu}$ . These are

$$\begin{aligned} r^4 g_1'(r) + 3r^3 g_1(r) + i\omega r^2 f_1(r) + ik g_2(r) &= 0, \\ -r^2 f_2'(r) - 2r f_2(r) + ik g_3(r) + i\omega g_1(r) + r(f_1(r) + f_3(r)) &= 0, \\ r^4 g_3'(r) + 3r^3 g_3(r) - ik r^2 f_3(r) + i\omega g_2(r) &= 0, \end{aligned} \quad (3.17)$$

so we can write  $f_1(r), f_3(r)$  and  $g_2(r)$  in terms of  $g_1(r), g_3(r)$  and  $f_2(r)$  and their derivatives. On the other hand we have  $\bar{g}^{\mu\nu} \delta T_{\mu\nu}^{NMG} = 0$  which together with the harmonic gauge condition implies traceless condition for metric fluctuations. Therefore we have  $f_1(r) + f_2(r) + f_3(r) = 0$ . This constraint fixes another function and one finds

$$g_3(r) = \frac{1}{ik} (r^2 f_2'(r) + 3r f_2(r) - i\omega g_1(r)). \quad (3.18)$$

If in equations of motion we impose the gauge condition (3.17) and constraint (3.18) then we will find six differential equations for two unknown functions  $g_1(r)$  and  $f_2(r)$ . Among these equations the  $rr$  component is a differential equation for  $f_2(r)$

$$\begin{aligned} 2r^8 f_2^{(4)} + 36r^7 f_2''' + ((191 - 2m^2 l^2) r^6 - 4(k^2 - \omega^2) r^4) f_2'' + ((-14m^2 l^2 + 329) r^5 + 20(\omega^2 - k^2) r^3) f_2' \\ + 2((-8m^2 l^2 + 68) r^4 + (m^2 l^2 - \frac{17}{2})(k^2 - \omega^2) r^2 + (k^2 - \omega^2)^2) f_2 = 0. \end{aligned} \quad (3.19)$$

In order to analysis this equation and find its asymptotic behavior we use the Frobenius's method. Let's insert a series solution  $f_2(r) = r^{-B} \sum_{n=0}^{\infty} \frac{c_n}{r^n}$  into the above equation and then take the  $r \rightarrow \infty$  limit and insert the coefficient of the greatest power of  $r$  to zero. In this way the possible values for  $B$  are ( $c_0 \neq 0$ )

$$B = 2, 4, 3 \pm \frac{1}{2} \sqrt{2 + 4m^2 l^2}. \quad (3.20)$$

What about the other unknown function  $g_1(r)$ ? We observed that one cannot write a linear combination of equations of motions in order to find a differential equation for  $g_1(r)$ . We need to consider higher order derivatives of equations of motion as well. After doing some calculations we will find an 8th order differential equation for  $g_1(r)$ . Since it is a very lengthy equation we have not write it and just present the final results. The asymptotic behavior of this differential equation can be found again by assuming  $g_1(r) = r^{-B'} \sum_{n=0}^{\infty} \frac{c'_n}{r^n}$ . The values of  $B'$  are

$$B' = 1, 3, 3, 5, 2 \pm \frac{1}{2} \sqrt{2 + 4m^2 l^2}, 4 \pm \frac{1}{2} \sqrt{2 + 4m^2 l^2}. \quad (3.21)$$

It is possible to find a relation between  $f_2(r)$  and derivatives of  $g_1(r)$ . By this relation one can verify, which values of  $B'$  are related to values of  $B$ . We find that for every value of  $B$  there are two values for  $B'$  in such a way that

$$B' = B \pm 1. \quad (3.22)$$

Again we demand that the asymptotic metric fluctuations fall off faster than the background metric. This will be happened if  $B$  and  $B' > 0$ .

Consider  $B' = B + 1$  then when  $B > 0$  automatically  $B' > 0$ . In this case all solutions of  $B$  are acceptable without any constraint except one. The only constraint happens when  $B = 3 - \frac{1}{2}\sqrt{2 + 4m^2l^2}$  which is acceptable for  $\frac{1}{2} < m^2l^2 < \frac{17}{2}$ .

When  $B' = B - 1$  then  $B > 1$ . Again all solutions are acceptable and the only constrain is coming from  $B = 3 - \frac{1}{2}\sqrt{2 + 4m^2l^2}$  where  $\frac{1}{2} < m^2l^2 < \frac{7}{2}$ .

### 3.2 $h_{\mu\varphi} = 0$ gauge

We are interested to know the behavior of the theory under another gauge fixing condition. So we do all steps in previous section but now for a new gauge  $h_{\mu\varphi} = 0$ . In this case it seems that the equations of motion cannot be written in the decoupled form (3.5). Despite this, we can study the solutions similar to previous sections.

#### 3.2.1 Highest weight solutions

The metric perturbations ansatz are given in (3.10). The highest weight equation gives the following functions (note that we have not traceless condition in this gauge)

$$\begin{aligned} f_1(r) &= -(C_2 + iC_4)r^2 - \frac{i(C_3 - C_6 + C_5 + C_4)}{1 + r^2} + C_1, \\ f_2(r) &= \frac{1}{(1 + r^2)}((C_1 + C_2 - iC_6 + 2iC_4 - iC_3)r^4 + (3iC_4 - 2iC_6 + C_2 + C_1)r^2 - iC_5), \\ f_3(r) &= r^2C_2 + iC_4r^2 + 3iC_4 + C_2 + C_1 + \frac{iC_5}{r^2}, \quad g_1(r) = \frac{C_4r^4 + (2C_4 + C_3)r^2 + C_6 - C_5}{r(1 + r^2)}, \\ g_2(r) &= (C_2 + iC_4)r^4 + (2iC_4 + C_2)r^2 - iC_5 + iC_6, \quad g_3(r) = \frac{C_4r^4 + C_5 + C_6r^2}{r(1 + r^2)}. \end{aligned} \quad (3.23)$$

To impose the gauge condition  $h_{\mu\varphi} = 0$  in the linearized equations of motions we have to choose  $C_1 = C_2 = C_4 = C_5 = C_6 = 0$ . Now there are six equations for just one unknown constant  $C_3$ . The only consistent solution to these equations is  $C_3 = 0$ . Such a solution is called the pure gauge solution. In other word, to have a highest weight solution with nontrivial values for frequency and wave number in this new gauge, we must have  $f_i(r) = g_i(r) = 0$  for  $i = 1, 2, 3$ . The same behavior previously has been observed in [21] for BTZ black holes.

#### 3.2.2 Asymptotic behavior of propagating solutions

Although we showed that the highest weight solution in the new gauge  $h_{\mu\varphi} = 0$  is a pure gauge, but let us try to find the asymptotic behavior of metric perturbations in this gauge. We start from the following ansatz

$$h_{\mu\nu} = l^2 e^{-i(\omega t - k\varphi)} \begin{pmatrix} (1 + r^2)f_1(r) & g_1(r) & 0 \\ g_1(r) & -(1 + r^2)^{-1}f_2(r) & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (3.24)$$

To find an asymptotic solution, we insert the above ansatz into the equations of motion (to simplify the calculations we have ignored 1 in  $1 + r^2$ , the final results are exactly the same). There are five independent



equations of motion which are not enough to decompose the differential equations. On the other hand we have  $\bar{g}^{\mu\nu}\delta T_{\mu\nu}^{NMG} = 0$  which implies  $\delta R = 0$ . If we use this equation together with its first and second derivatives then we can obtain a differential equation for  $f_1(r)$ . If insert  $ml = \xi$  then for  $f_1(r)$  we find the following differential equation

$$\begin{aligned} & \left(\frac{1}{2} + \xi^2\right)r^2 - \omega^2 + k^2 \Big) r^{11} f_1^{(5)} + \left(9(1+2\xi^2)r^4 + \left(\frac{41}{2} + \xi^2\right)k^2 - 20\omega^2\right)r^2 - \omega^2 k^2 + k^4 \Big) r^8 f_1^{(4)} + \left((47+93\xi^2-2\xi^4)r^4 \right. \\ & + (2(59+\xi^2)k^2 + (4\xi^2-115)\omega^2)r^2 - 4\omega^2 k^2 + 6k^4 - 2\omega^4 \Big) r^7 f_1''' - 2r^4 \left((7\xi^4 - \frac{147}{2}\xi^2 - \frac{77}{2})r^6 + \left(\xi^4 + \frac{23}{2}\xi^2 - 104\right)k^2 \right. \\ & + \omega^2(103-13\xi^2))r^4 + 2\left(\frac{5}{4} + \xi^2\right)k^2 - 3\omega^2 \Big) (k^2 - \omega^2)r^2 + k^2(k^2 - \omega^2)^2 \Big) f_1'' + r^3 \left((\xi^6 - \frac{37}{2}\xi^4 + \frac{179}{4}\xi^2 + \frac{217}{8})r^6 + \left(\xi^4 \right. \right. \\ & - 36\xi^2 + \frac{311}{4})k^2 - 3(\xi^2 - \frac{15}{2})(\xi^2 - \frac{7}{2})\omega^2)r^4 + \left((\xi^2 - \frac{29}{2})k^2 - 3(\xi^2 - \frac{7}{2})\omega^2\right)(k^2 - \omega^2)r^2 + (k^2 - \omega^2)^3 \Big) f_1' + \left((k^2 - \omega^2)^3 \right. \\ & + (\xi^2 - \frac{1}{2})^2(\frac{1}{2} + \xi^2)r^6 + 3(\xi^2 - \frac{1}{2})^2(k^2 - \omega^2)r^4 + 3(\frac{1}{2} + \xi^2)(k^2 - \omega^2)^2 r^2 \Big) k^2 f_1 = 0. \end{aligned} \quad (3.25)$$

If we consider an asymptotic series solution for the above equation, i.e.  $f_1(r) = r^{-B} \sum_{n=0}^{\infty} \frac{a_n}{r^n}$ , then we can read the values of  $B$  from equation  $B(B^2 - 6B + \frac{17}{2} - \xi^2)(B^2 - 2B + \frac{1}{2} - \xi^2) = 0$ . So the possible values of  $B$  are

$$B = 0, B = 1 \pm \frac{1}{2}\sqrt{4\xi^2 + 2}, B = 3 \pm \frac{1}{2}\sqrt{4\xi^2 + 2}. \quad (3.26)$$

Doing the same computation but for  $f_2(r) = r^{-B'} \sum_{n=0}^{\infty} \frac{a'_n}{r^n}$  gives another complicated differential equation. The final result for values of  $B'$  will be

$$B' = 2, 1 \pm \frac{1}{2}\sqrt{4\xi^2 + 2}. \quad (3.27)$$

For  $g_1(r) = r^{-B''} \sum_{n=0}^{\infty} \frac{a''_n}{r^n}$  we find the following values of  $B''$

$$B'' = 1, B'' = 2 \pm \frac{1}{2}\sqrt{4\xi^2 + 2}, B'' = \pm \frac{1}{2}\sqrt{4\xi^2 + 2}. \quad (3.28)$$

To suppress the blow up of metric fluctuations in asymptotic region, we must limit power behaviors to  $B > 0$ ,  $B' > 0$  and  $B'' > 0$ .

## 4 Warped $AdS_3$ vacuum in NMG

Another vacuum solution to the equations of motion with less symmetries than the  $AdS_3$  background is the warped  $AdS_3$  vacuum

$$ds^2 = \frac{l^2}{\nu^2 + 3} \left[ -(1+r^2)d\tau^2 + \frac{dr^2}{1+r^2} + \frac{4\nu^2}{\nu^2 + 3}(d\varphi + r d\tau)^2 \right]. \quad (4.1)$$

We define the warp factor as

$$\sigma = \frac{2\nu}{\sqrt{\nu^2 + 3}}. \quad (4.2)$$

The generator of time translations is given by a Killing vector as  $(\frac{l^2}{\nu^2+3}[-1+3(\frac{\nu^2-1}{\nu^2+3})r^2])^{-\frac{1}{2}}\partial_\tau$ . For  $\nu^2 < 1$  it is always time-like but for  $\nu^2 > 1$  there is a transition surface from time-like to space-like. For  $0 \leq \nu^2 < 1$  the

warp factor  $0 \leq \sigma < 1$  and the space-time is called squashed. For  $\nu^2 > 1$  we have  $1 \leq \sigma < 2$  and it is called stretched. The special case  $\nu = \sigma = 1$  corresponds to a deformed  $AdS_3$  with a fibration [13].

The isometry group of  $AdS_3$ , i.e.  $SL(2, R)_L \times SL(2, R)_R$ , is broken to  $SL(2, R)_L \times U(1)_R$  due to presence of the warp factor  $\sigma$ . The generators of this symmetry transformation can be constructed out of the Killing vectors and are given by

$$L_0 = i\partial_\tau, \quad L_{\pm 1} = \pm e^{\pm i\tau} \left( \frac{r}{\sqrt{1+r^2}} \partial_\tau \mp i\sqrt{1+r^2} \partial_r + \frac{1}{\sqrt{1+r^2}} \partial_\varphi \right); \quad J = -i\partial_\varphi. \quad (4.3)$$

If we consider the NMG Lagrangian (2.1) and insert the vacuum solution (4.1) into equations of motion (2.2) then we will find the following values for NMG mass parameter  $m$  and cosmological constant  $\Lambda$  in terms of the warp factor  $\sigma$  and the warped  $AdS_3$  scalar curvature ( $R = -\frac{6}{l^2}$ )

$$m^2 = -\frac{3}{2l^2} \frac{21\sigma^2 - 4}{\sigma^2 - 4}, \quad \Lambda = -\frac{3}{2l^2} \frac{21\sigma^4 - 72\sigma^2 + 16}{21\sigma^4 - 88\sigma^2 + 16}. \quad (4.4)$$

Since the warped factor is limited between  $0 \leq \sigma < 2$  we can specify the sign of parameters of the theory for different values of the warped factor. Supposing  $l^2 > 0$  (or negative curvature) then we have the following behaviors for mass parameter and cosmological constant

$\sigma$	$0 \leq \sigma < \sigma_1$	$\sigma_1 < \sigma < \sigma_2$	$\sigma_2 < \sigma < \sigma_3$	$\sigma_3 < \sigma < 2$
$m^2$	$< 0$	$> 0$	$> 0$	$> 0$
$\Lambda$	$< 0$	$> 0$	$< 0$	$> 0$

Table 1: Behavior of mass parameter and cosmological constant in NMG for  $0 \leq \sigma < 2$ .

The critical values in warped factor which have been appeared in the above table are

$$\sigma_1 = \frac{2}{\sqrt{21}} \cong 0.436, \quad \sigma_2 = \frac{2\sqrt{189 - 42\sqrt{15}}}{21} \cong 0.489, \quad \sigma_3 = \frac{2\sqrt{189 + 42\sqrt{15}}}{21} \cong 1.786. \quad (4.5)$$

In what follows we will find different behaviors of metric fluctuations around the warped  $AdS_3$  metric for different gauge conditions.

## 4.1 Harmonic Gauge

As we mentioned before since the warped  $AdS_3$  background has less symmetries than  $AdS_3$  we do not have the simple rules as equation (3.4). This makes the equations more complicated, so in most cases we just present the final or important results.

### 4.1.1 Highest weight solutions

Unlike the  $AdS_3$  vacuum, here we consider the behavior of metric fluctuation only in presence of the harmonic gauge condition  $\nabla_\mu h^{\mu\nu} = 0$  and we have not a traceless condition. The highest weight condition for metric perturbations are given by the following relations

$$J h_{\mu\nu} = k h_{\mu\nu}, \quad L_0 h_{\mu\nu} = \omega h_{\mu\nu}, \quad L_1 h_{\mu\nu} = 0, \quad (4.6)$$

where  $L_{\pm 1}, L_0, J$  are Killing vectors that generate the  $SL(2, R)_R \times U(1)_L$  isometry group which are given in (4.3). Like the previous case we choose the following ansatz

$$h_{\mu\nu}(\tau, r, \varphi) = f_6(r) e^{i(k\varphi - \omega\tau)} \begin{pmatrix} f_1(r) & f_2(r) & f_3(r) \\ f_2(r) & f_4(r) & f_5(r) \\ f_3(r) & f_5(r) & C_6 \end{pmatrix}. \quad (4.7)$$

The (3, 3) component of the highest weight condition  $L_1 h_{\mu\nu} = 0$  fixes the value of  $f_6(r)$  to

$$f_6(r) = e^{k \tan^{-1} r} (1 + r^2)^{-\frac{\omega}{2}}. \quad (4.8)$$

Using this value, and by solving differential equations from other components of  $L_1 h_{\mu\nu} = 0$  one finds the following solutions

$$\begin{aligned} f_1(r) &= (C_6 - C_3 - 2iC_2)r^2 + (C_4 + iC_1)r - C_5, & f_2(r) &= \frac{(2C_1 - iC_4)r^2 + 2(C_2 + iC_5 - iC_3)r + iC_4}{2(1 + r^2)}, \\ f_3(r) &= (C_6 - iC_2)r + iC_1, & f_4(r) &= \frac{C_5 r^2 + C_4 r + C_3}{(1 + r^2)^2}, & f_5(r) &= \frac{C_1 r + C_2}{1 + r^2}. \end{aligned} \quad (4.9)$$

In the case of harmonic gauge we can fix three of the  $C_i$ 's in terms of other constants, i.e.

$$\begin{aligned} C_1 &= \frac{i\sigma^2}{2(k^2 + \sigma^4)} \left( C_4 k^2 + 2((\sigma^2 - 1)C_3 + (1 - \omega)C_5)k + \sigma^2 C_4 (2 - \omega) \right), \\ C_2 &= \frac{i\sigma^2}{2(k^2 + \sigma^4)} \left( 2C_3 k^2 - C_4 (\sigma^2 - (2 - \omega))k + 2(C_3 - (1 - \omega)C_5)\sigma^2 \right), \\ C_6 &= \frac{1}{2k(k^2 + \sigma^4)} \left( -2C_3 k^3 + (\sigma^2 + 2\omega - 3)C_4 k^2 + \left( ((-2C_5 + 2C_3)\omega + 2C_5 - 4C_3)\sigma^2 \right. \right. \\ &\quad \left. \left. + 2(1 - \omega)((\omega - 1)C_5 + C_3) \right) k - \sigma^2 C_4 (1 - \omega)(2 - \omega) \right) \sigma^4. \end{aligned} \quad (4.10)$$

We now substitute the ansatz (4.7) in the linearized equations of motion and use the above gauge conditions. To have a non-trivial solution, the determinant of coefficients of  $C_3, C_4$  and  $C_5$  for each subset of equations must be zero. Independent of choice of subsets we find always two polynomials  $P_1$  and  $P_2$ . Defining  $\omega = \pm u^{\frac{1}{2}} + \frac{1}{2}$

$$\begin{aligned} P_1 &= \frac{1}{64} (2(16k^4 - 136k^2 + 9)\sigma^8 + (64k^6 - 368k^4 + 652k^2 - 45)\sigma^6 - 4k^2(48k^4 - 160k^2 + 95)\sigma^4 \\ &\quad + 16k^4(12k^2 - 19)\sigma^2 - 64k^6) + \frac{1}{16} (4(4k^2 - 5)\sigma^8 + (48k^4 - 152k^2 + 59)\sigma^6 - 8k^2(12k^2 - 17)\sigma^4 \\ &\quad + 48k^4\sigma^2)u + \left( \frac{1}{2}\sigma^8 + \frac{3}{4}(4k^2 - 5)\sigma^6 - 3k^2\sigma^4 \right) u^2 + \sigma^6 u^3 = 0, \\ P_2 &= 84\sigma^8 + 24(2k^2 - 5)\sigma^6 - \frac{1}{16} (176k^4 + 424k^2 - 333)\sigma^4 + \frac{1}{8} (176k^4 - 208k^2 + 9)\sigma^2 + (11k^4 + \frac{9}{2})k^2 \\ &\quad + (\sigma^2(48\sigma^4 + \frac{31}{2}\sigma^2 - 5) - 2k^2(11\sigma^4 - 12\sigma^2 + 1))u - \sigma^2(11\sigma^2 - 2)u^2 = 0. \end{aligned} \quad (4.11)$$

Solving these equations gives the frequencies of allowed modes. We will discuss these modes in the next sections.

#### 4.1.2 Asymptotic behavior of propagating solutions

In order to find the behavior of metric perturbations at the asymptotic limit as a propagating mode let us consider the following ansatz

$$h_{\mu\nu}(\tau, r, \varphi) = \frac{l^2(4 - \sigma^2)}{12} e^{-i(\omega t - k\varphi)} \begin{pmatrix} -(1 + r^2)f_1(r) & \sigma^2 g_1(r) & r\sigma^2 g_2(r) \\ \sigma^2 g_1(r) & (1 + r^2)^{-1}f_2(r) & \sigma^2 g_3(r) \\ r\sigma^2 g_2(r) & \sigma^2 g_3(r) & -\sigma^2 f_3(r) \end{pmatrix}. \quad (4.12)$$

If we impose the Harmonic gauge we will find three equations between unknown functions as

$$\begin{aligned}
(1+r^2)^2 f_2' - i(1+r^2) \left( (k(\sigma^2 - 1)r^2 + \sigma^2 \omega r - k)g_3 - \sigma^2(kr + \omega)g_1 - ir(f_1 - f_2) \right) - r\sigma^2((r^2 - 1)g_2 - f_3) &= 0, \\
\sigma^2(1+r^2)^2 g_1' - i(1+r^2)(kr + \omega)f_1 - r \left( i((\sigma^2 - 1)kr^2 + \sigma^2 \omega r - k)g_2 - 2\sigma^2(1+r^2)g_1 \right) &= 0, \\
\sigma^2(1+r^2)^2 g_3' + i((kr + \omega)r\sigma^2 - k(1+r^2))f_3 + \sigma^2 r(i(kr + \omega)g_2 + 2(r^2 + 1)g_3) &= 0.
\end{aligned} \tag{4.13}$$

If we find  $f_1(r)$ ,  $f_3(r)$  and  $g_2(r)$  from above gauge conditions and insert them into the equations of motion we will find six mixed differential equations for three remaining functions. To find the asymptotic  $r \rightarrow \infty$  behavior of the solutions there are two approaches. In first approach similar to previous cases, we recombine equations of motion and try to find a decoupled differential equation for each unknown function. In the second approach, we can use the following behaviors for the remaining functions

$$g_1(r) \rightarrow C_1 r^{-B}, \quad f_2(r) \rightarrow C_2 r^{-B}, \quad g_3(r) \rightarrow C_3 r^{-B-1}. \tag{4.14}$$

By putting these relations into the equations of motion and by going to the asymptotic region one finds six equations for three unknown constants  $C_1, C_2$  and  $C_3$ . To find  $B$ , it is enough to select three out of six equations and insert the determinant of the coefficients to zero. These values of  $B$  again must be independent of the choice of equations. Doing all steps above, we will find again two polynomials from highest weight approach (4.11) just by replacing  $\omega$  by  $B$ .

## 4.2 $h_{\mu\varphi} = 0$ Gauge

The calculations in this section is roughly analogous to one accomplished for TMG in [15] while we do it here for NMG.

### 4.2.1 Highest weight solutions

The situation here is very similar to the previous case. We start from highest weight conditions in (4.6) and find exactly the same value for  $f_6(r)$  as (4.8). Substituting it in the remaining equations we will obtain other functions as

$$\begin{aligned}
f_1(r) &= (C_1 - C_6)r^2 + 3C_2r - C_3, \quad f_2(r) = \frac{3C_2r^2 + 2(C_6 - C_3)r - C_2}{2i(1+r^2)}, \\
f_3(r) &= C_1r + C_2, \quad f_4(r) = \frac{C_3r^2 + C_2r + C_1}{(1+r^2)^2}, \quad f_5(r) = \frac{C_2r - C_1 + C_6}{i(1+r^2)}.
\end{aligned} \tag{4.15}$$

Next, we use the gauge fixing condition  $h_{\mu\varphi} = 0$  such that the highest weight condition is preserved. We choose  $C_6 = f_3(r) = f_5(r) = 0$  which states that  $C_1 = C_2 = 0$ . We now insert this perturbations into the linearized equations of motion. We can show that for each subset of equations of motion we have a matrix

$$\mathcal{M}_{3 \times 3} \begin{pmatrix} C_3 \\ C_4 \\ C_5 \end{pmatrix} = 0, \tag{4.16}$$

where the propagating modes are those with  $\det \mathcal{M} = 0$  while the pure gauge modes obtain by  $\det \mathcal{M} \neq 0$  or equivalently  $C_3 = C_4 = C_5 = 0$ . There is a unique factor in all determinants of  $\mathcal{M}$  for all subset of equations and we find that  $\det \mathcal{M} \sim P_2$ . So in this gauge just one of the previous polynomials survives. This polynomial gives the value of possible frequencies for each propagating mode in the highest weight

$$\begin{aligned}\omega_{1,2} &= \frac{1}{2} + \frac{1}{\sqrt{11\sigma^2 - 2}} \left[ 96\sigma^6 + (31 - 44k^2)\sigma^4 - (10 - 48k^2)\sigma^2 - 4k^2 \pm 4 \left( 1500\sigma^{12} - 1116\sigma^{10} \right. \right. \\ &\quad \left. \left. + (409 + 18k^2)\sigma^8 - (68 + 40k^2)\sigma^6 + (4 + 26k^2 + k^4)\sigma^4 - 2(2 + k^2)k^2\sigma^2 + k^4 \right)^{\frac{1}{2}} \right]^{\frac{1}{2}}, \\ \omega_{3,4} &= \frac{1}{2} - \frac{1}{\sqrt{11\sigma^2 - 2}} \left[ 96\sigma^6 + (31 - 44k^2)\sigma^4 - (10 - 48k^2)\sigma^2 - 4k^2 \pm 4 \left( 1500\sigma^{12} - 1116\sigma^{10} \right. \right. \\ &\quad \left. \left. + (409 + 18k^2)\sigma^8 - (68 + 40k^2)\sigma^6 + (4 + 26k^2 + k^4)\sigma^4 - 2(2 + k^2)k^2\sigma^2 + k^4 \right)^{\frac{1}{2}} \right]^{\frac{1}{2}}.\end{aligned}\quad (4.17)$$

#### 4.2.2 Asymptotic behavior of propagating solutions

To find the behavior of metric perturbations in  $h_{\mu\varphi} = 0$  gauge let's consider the following ansatz

$$h_{\mu\nu} = e^{-i(\omega t - k\varphi)} \begin{pmatrix} -(1+r^2)f_1(r) & f_2(r) & 0 \\ f_2(r) & (1+r^2)^{-1}f_3(r) & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (4.18)$$

As before, we have six equations of motion where just five of them are independent. On the other hand we can use the trace of energy momentum tensor  $\bar{g}^{\mu\nu}\delta T_{\mu\nu} = 0$ . If we use this equation together with its first and second derivatives we will have eight equations. After a little computation we can find decoupled differential equations for  $f_1(r)$ ,  $f_2(r)$  or  $f_3(r)$ . For each function if we write an asymptotic expansion series, we will find the same behavior. The fall off power again is given by a polynomial,  $B = \pm u^{\frac{1}{2}} + \frac{1}{2}$

$$\begin{aligned}P_3 &= 84\sigma^8 + 24(2k^2 - 5)\sigma^6 - \frac{1}{16}(176k^4 + 424k^2 - 333)\sigma^4 + \frac{1}{8}(176k^4 - 208k^2 + 9)\sigma^2 + (11k^4 + \frac{9}{2})k^2 \\ &+ (\sigma^2(48\sigma^4 + \frac{31}{2}\sigma^2 - 5) - 2k^2(11\sigma^4 - 12\sigma^2 + 1))u - \sigma^2(11\sigma^2 - 2)u^2 = 0.\end{aligned}\quad (4.19)$$

As we see  $P_3$  is equal to  $P_2$  in the previous gauge (4.11). As another equivalent approach to find the asymptotic  $r \rightarrow \infty$  of the solutions, we can insert the following behaviors for fluctuation functions

$$f_1(r) \rightarrow C_1 r^{-B}, \quad f_2(r) \rightarrow C_2 r^{-B}, \quad f_3(r) \rightarrow C_3 r^{-B}. \quad (4.20)$$

By inserting these values into the equations of motion and by going to the large values of  $r$  one finds six Algebraic equations for three unknown constants  $C_1, C_2$  and  $C_3$ . The only consistent nontrivial solution happens when we have the  $P_3$  polynomial for values of  $B$ .

## 5 Stability

There are two main conditions that must be checked in order to have a stable solution. The first one is the positivity of the energy of a typical solution and the second one is the reality condition for frequencies. In this section we perform both these checks to find the domain of reliability of our solutions.

## 5.1 Energy condition

To find the energy condition we follow the same approach presented in [15] for TMG (one may also use the ADT construction [23]). According to [24] the conserved charges  $Q(\xi)$  of a diffeomorphism invariant theory, which are associated with the Killing vectors  $\xi^\mu$ , are given by the following expression

$$Q(\xi) = \frac{1}{16\pi G} \int_{\Sigma} \star(\xi^\mu E_{\mu\nu}^{(2)}[h^{(1)}]dx^\nu), \quad (5.1)$$

where  $\Sigma$  is a spatial hyper-surface at constant time and  $\star$  represents the Hodge star operation. To find the conserved energies, we substitute the first order perturbations  $h_{\mu\nu}^{(1)}$  of the highest weight solutions, into the  $E_{\mu\nu}^{(2)}$ , the energy-momentum pseudo-tensor. The energy density of the gravitational wave is given by [15]

$$\mathcal{E} = \frac{1}{16\pi G} \int dr \sqrt{-\bar{g}} \bar{g}^{\tau\mu} E_{\mu\nu}^{(2)} \xi^\nu, \quad (5.2)$$

where we have used  $\xi^\mu = (1, 0, 0)$  to find the energy. We consider the physical perturbations in their real form

$$\psi_{\mu\nu} = \alpha h_{\mu\nu} + \alpha^* (h_{\mu\nu})^*, \quad (5.3)$$

where  $h_{\mu\nu}$ 's are given by (4.7)-(4.9) in the harmonic gauge and by (4.7) together with (4.15) in the other gauge. After applying the gauge conditions we can remove the coefficients  $C_1, C_2, C_6$  and write  $C_3, C_4$  in terms of  $C_5$  from linearized equations of motion, i.e.,

$$C_3 = -\frac{S_1(\omega, k, \sigma)}{S_0(\omega, k, \sigma)} C_5, \quad C_4 = \frac{\sigma^2}{k} \frac{S_2(\omega, k, \sigma)}{S_0(\omega, k, \sigma)} C_5, \quad (5.4)$$

where  $S_0, S_1$  and  $S_2$  are real functions and are given in the appendix A.

To avoid the divergences we consider the energy density per unit length in the  $\varphi$ -direction as in [15]. The final result for energy can be written as the following sum

$$\mathcal{E} = |\alpha C_5|^2 \sum_{n=0}^8 \left( \int_{-\infty}^{+\infty} dr \frac{r^n e^{2k \tan^{-1} r}}{(1+r^2)^{\omega+4}} \right) (B_n(k, \sigma)) \equiv \sum_{n=0}^8 A_n(k, \sigma) B_n(k, \sigma). \quad (5.5)$$

The above integrals are finite for  $Re(\omega) > \frac{1}{2}$  and for  $n \leq 8$  and they obey a recursion relation [15]

$$(2\omega + 7 - n)A_n = 2kA_{n-1} + (n-1)A_{n-2}, \quad (5.6)$$

this enables us to write the energy in terms of  $A_0$  which is real and positive. As we showed in previous sections there are two types of solutions labeled by two polynomials:

- **Massive modes:**

If we insert the  $P_2$  polynomial which exists in both  $h_{\mu\varphi} = 0$  and harmonic gauges we will find a negative energy density in  $0 \leq \sigma < 2$  and for all values of  $k$ . This negativity of energy for massive modes is similar to the TMG [15]. Therefore all massive modes in the squashed or stretched space-times are unstable.

- **Massless modes:**

Interestingly if we examine the energy density of modes corresponding to the  $P_1$  polynomial which exists only in the harmonic gauge, the result will be zero. So this means that these modes are describing the massless modes of the theory. For TMG in [15] since they have considered only the  $h_{\mu\varphi} = 0$  gauge they have not seen such massless modes. To find these modes in TMG we performed computations of [15] in the harmonic gauge. Again the results contains two type of polynomials. The massive modes found in the  $h_{\mu\varphi} = 0$  gauge and another mode. The new mode has exactly the same polynomial structure as  $P_1$  in NMG case and its energy density is zero.

## 5.2 Frequency condition

As we mentioned before another check for stability of solutions is the reality condition for frequencies. In order to do this we need to solve the polynomials and find their roots. These roots were the values of  $B$  (fall off powers for propagating solutions) or they were the allowed frequencies  $\omega$  for highest weight modes. We demand that these roots to be positive and real valued.

Since we used a change of variable as  $\omega = \pm u^{\frac{1}{2}} + \frac{1}{2}$  and because the energy condition restrict us to  $Re(\omega) > \frac{1}{2}$  so only the plus sign is allowed and the reality condition for frequencies translates to  $Re(u) > 0$ .

Although the  $P_2$  polynomial shows a negative energy modes but in what follows we will check the regions of real frequency for massive modes as well.

### 5.2.1 $P_2$ polynomial (Massive modes)

Starting from  $P_2$ , we find the following roots for  $u$

$$u_{\pm} = \frac{\Delta_1 \pm 4\Delta_2^{\frac{1}{2}}}{4\sigma^2(11\sigma^2 - 2)}, \quad \Delta_1 = \sigma^2(96\sigma^4 + 31\sigma^2 - 10) - 4(\sigma^2 - 1)(11\sigma^2 - 1)k^2, \\ \Delta_2 = \sigma^4(4 - 68\sigma^2 + 409\sigma^4 - 1116\sigma^6 + 1500\sigma^8) + 2\sigma^2(9\sigma^2 - 2)(\sigma^2 - 1)^2k^2 + (\sigma^2 - 1)^2k^4. \quad (5.7)$$

As we mentioned before, to have a real value for  $\omega$ , we must have positive real values for  $u_{\pm}$ . According to the roots of  $\Delta_2$  we have two situations:

- **When  $\Delta_2$  has two real roots for  $k^2$**

1. To have a real valued  $u_{\pm}$  we need  $\Delta_2 \geq 0$ . Suppose that solving  $\Delta_2 = 0$  in terms of  $k^2$  gives two real roots so we can write  $\Delta_2 = (\sigma^2 - 1)^2(k^2 - \Sigma_-)(k^2 - \Sigma_+)$ . So the reality condition of  $u_{\pm}$  will restrict  $k^2$  to (note that  $\Sigma_- > \Sigma_+$ )

$$0 \leq k^2 \leq \Sigma_+, \quad k^2 \geq \Sigma_-, \quad \Sigma_{\pm} = \sigma^2 \frac{(\sigma^2 - 1)(2 - 9\sigma^2) \pm \sigma(3(2 - 11\sigma^2)(43\sigma^4 - 20\sigma^2 + 4))^{\frac{1}{2}}}{\sigma^2 - 1}. \quad (5.8)$$

2. On the other hand  $\Delta_2$  itself must be real, which means that  $\Sigma_{\pm}$  must be real. This will be possible if we choose  $0 \leq \sigma \leq \sqrt{\frac{2}{11}}$  in the admissible interval of  $0 \leq \sigma < 2$ .

3. One must notice that in the interval  $0 \leq \sigma \leq \sqrt{\frac{2}{11}} \cong 0.426$  we have always  $\Sigma_- \geq 0$ , but for  $0.381 \leq \sigma \leq 0.398$  we have  $\Sigma_+ < 0$ . So in this sub-interval we have just  $k^2 \geq \Sigma_-$ .

4. By looking to the values of  $u_{\pm}$ , in the interval  $0 \leq \sigma \leq \sqrt{\frac{2}{11}}$  we see that the denominator has a negative value for both  $u_{\pm}$ . So  $u_{\pm} \geq 0$  if  $\Delta_1 \pm 4\Delta_2^{\frac{1}{2}} \leq 0$ . In this interval  $\Delta_1$  changes sign from negative values to positive values and it is a monotonically increasing function for each values of  $k^2$ , and  $-4k^2 \leq \Delta_1$ . Also  $4\Delta_2^{\frac{1}{2}}$  is a positive monotonically decreasing function for each values of  $k^2$  and  $4\Delta_2^{\frac{1}{2}} \leq 4k^2$ .

5. Finally we observe that  $|4\Delta_2^{\frac{1}{2}}| > |\Delta_1|$  for all values in the mentioned interval and for all  $k^2$ , so we conclude that just  $u_-$  is an acceptable solution.

• **When  $\Delta_2$  has not real roots for  $k^2$**

1. In this case we are restricted to  $\sqrt{\frac{2}{11}} \leq \sigma < 2$  where  $\Delta_2$  has not any real roots and we have  $\Delta_2 \geq 0$ . So for  $u_{\pm} \geq 0$  we need  $\Delta_1 \pm 4\Delta_2^{\frac{1}{2}} \geq 0$ .

2. In  $\Delta_1 = \sigma^2(96\sigma^4 + 31\sigma^2 - 10) - 4(\sigma^2 - 1)(11\sigma^2 - 1)k^2$ , the first term (coefficient of  $k^0$ ) is always positive in  $\sqrt{\frac{2}{11}} \leq \sigma < 2$  but the coefficient of  $k^2$  changes sign from positive values to negative values at  $\sigma = 1$ , so  $\Delta_1$  can be either positive or negative.

3. If  $|\Delta_1| < 4\Delta_2^{\frac{1}{2}}$  or equivalently  $\Delta_1^2 - 16\Delta_2 < 0$  then  $u_+$  is allowed but  $u_-$  is not. On the other hand if  $|\Delta_1| > 4\Delta_2^{\frac{1}{2}}$  or equivalently  $\Delta_1^2 - 16\Delta_2 > 0$  then we have two choices. If  $\Delta_1 > 0$  then both  $u_{\pm}$  are allowed and if  $\Delta_1 < 0$  then neither  $u_+$  nor  $u_-$  are not allowed.

4. Consider  $\Delta_1^2 - 16\Delta_2 > 0$ . If  $\sqrt{\frac{2}{11}} < \sigma < 1$  then always  $\Delta_1 > 0$  so

$$\Xi_0 < k^2, \quad \Xi_0 = \frac{\sigma^2(96\sigma^4 + 31\sigma^2 - 10)}{4(\sigma^2 - 1)(11\sigma^2 - 1)}, \quad (5.9)$$

but in this interval  $\Xi_0 < 0$  so  $k^2 > 0$  and therefore there is no restriction here. But

$$\Delta_1^2 - 16\Delta_2 = \sigma^2(11\sigma^2 - 2)(-3\sigma^2(448\sigma^6 - 640\sigma^4 + 111\sigma^2 + 6) - 8(\sigma^2 - 1)(96\sigma^4 + 43\sigma^2 - 9)k^2 + 176(\sigma^2 - 1)^2k^4), \quad (5.10)$$

which by  $\Delta_1^2 - 16\Delta_2 > 0$  assumption  $k^2$  restricts to

$$k^2 < \Xi_+, \quad k^2 > \Xi_-, \quad \Xi_{\pm} = \frac{1}{44(\sigma^2 - 1)}(96\sigma^4 + 43\sigma^2 - 9 \pm (81 - 576\sigma^2 + 3784\sigma^4 + 24000\sigma^8 - 12864\sigma^6)^{\frac{1}{2}}). \quad (5.11)$$

In this interval we find that  $\Xi_+ < \Xi_0 < \Xi_-$  so  $\Xi_+ < 0$  therefore  $k^2 < \Xi_+$  is not allowed. If we look at  $\Xi_-$  we see that it changes sign from positive values to negative values at  $\sigma = 0.506$ . The final result is as follows:

$$\sqrt{\frac{2}{11}} < \sigma < 0.506 \rightarrow \Xi_- > 0 \rightarrow k^2 > \Xi_-, \quad 0.506 < \sigma < 1 \rightarrow \Xi_- < 0 \rightarrow k^2 > 0, \quad (5.12)$$

and two solutions  $u_{\pm}$  are allowed.

5. Consider  $\Delta_1^2 - 16\Delta_2 > 0$ . If  $1 < \sigma < 2$  and  $\Delta_1 > 0$  then  $\Xi_0 > k^2$ . In this interval of  $\sigma$  we have always  $\Xi_- < \Xi_0 < \Xi_+$ . Here  $\Xi_0 > 0$  and  $\Xi_+ > 0$  but  $\Xi_-$  changes sign from positive values to negative values at  $\sigma = 1.103$ . So two solutions  $u_{\pm}$  are allowed, either when  $k^2 > \Xi_+$  in all the interval or if  $k^2 < \Xi_-$  when  $1 < \sigma < 1.103$ .



6. Consider  $\Delta_1^2 - 16\Delta_2 < 0$  where just  $u_+$  was allowed. For  $\sqrt{\frac{2}{11}} < \sigma < 1$  we find  $\Xi_+ < k^2 < \Xi_-$  but since in this interval  $\Xi_+ < 0$  and  $\Xi_-$  changes sign we conclude that for  $\sqrt{\frac{2}{11}} < \sigma < 0.506$  we have  $0 < k^2 < \Xi_-$ . For  $1 < \sigma < 2$  we find  $\Xi_- < k^2 < \Xi_+$ . Here  $\Xi_+ > 0$  and  $\Xi_-$  changes sign so for  $1 < \sigma < 1.103$  we have  $\Xi_- < k^2 < \Xi_+$  and for  $1.103 < \sigma < 2$  we have  $0 < k^2 < \Xi_+$ .

### 5.2.2 $P_1$ polynomial (Massless modes)

The  $P_1$  polynomial was  $au^3 + bu^2 + cu + d = 0$  where

$$\begin{aligned} a &= \sigma^6, \quad b = \sigma^4(3(\sigma^2 - 1)k^2 + \frac{1}{4}\sigma^2(2\sigma^2 - 15)), \\ c &= \sigma^2(3(\sigma^2 - 1)^2k^4 + \frac{1}{2}\sigma^2(\sigma^2 - 1)(2\sigma^2 - 17)k^2 - \frac{1}{16}\sigma^4(20\sigma^2 - 59)), \\ d &= (\sigma^2 - 1)^3k^6 + \frac{1}{4}(\sigma^2 - 1)^2(2\sigma^2 - 19)\sigma^2k^4 - \frac{1}{16}\sigma^4(\sigma^2 - 1)(68\sigma^2 - 95)k^2 + \frac{9}{64}\sigma^6(2\sigma^2 - 5). \end{aligned} \quad (5.13)$$

As we see from these coefficients for  $0 \leq \sigma \leq 1$  we have always  $a > 0$ ,  $b < 0$ ,  $c > 0$  and  $d < 0$ . According to Descartes' rule of signs this polynomial has either three or one real positive root. So at least there is one real positive root in  $0 \leq \sigma \leq 1$  for all values of  $k^2$ .

Dividing  $P_1$  by  $\sigma^6$  and changing variable as  $u = t + \frac{5}{4} - \frac{\sigma^2}{6} - \frac{\sigma^2 - 1}{\sigma^2}k^2$  we can write the polynomial in its depressed form,  $t^3 + pt + q = 0$

$$P_1 = t^3 - \frac{\sigma^2(\sigma^4 + 12) + 12(\sigma^2 - 1)k^2}{12\sigma^2}t - \frac{-\sigma^4(\sigma^4 - 36) + 18(\sigma^2 - 1)(17\sigma^2 - 6)k^2}{108\sigma^2} = 0. \quad (5.14)$$

The roots of this equation depend on the sign of its discriminant  $\Delta = \frac{q^2}{4} + \frac{p^3}{27}$ . If  $\Delta > 0$  then there is just one real root,  $t = (-\frac{q}{2} + \Delta^{\frac{1}{2}})^{\frac{1}{3}} + (-\frac{q}{2} - \Delta^{\frac{1}{2}})^{\frac{1}{3}}$  and if  $\Delta \leq 0$  then there are three real roots for  $t$ . The discriminant is

$$\Delta = -\frac{(\sigma^4 - 4)^2}{432} - \frac{(\sigma^2 - 1)(3\sigma^8 - \sigma^6 - 98\sigma^4 + 36\sigma^2 + 24)}{216\sigma^2}k^2 + \frac{(863\sigma^4 - 612\sigma^2 + 60)(\sigma^2 - 1)^2}{432\sigma^4}k^4 - \frac{(\sigma^2 - 1)^3}{27\sigma^6}k^6. \quad (5.15)$$

It is very hard to find exactly where we have a change sign in  $\Delta$  but since  $\sigma$  is limited between 0 and 2 we can compare values of  $k$  with  $\sigma$  in three regions.

•  $k \rightarrow 0$

In this case only the first term in  $\Delta$  is dominant ( $\Delta \cong -\frac{(\sigma^4 - 4)^2}{432}$ ) so for all values of  $\sigma$  its value is negative and we have three roots for  $t$ . In this region  $u \cong t + \frac{5}{4} + \frac{\sigma^2}{6}$  and  $t$  has three positive roots  $t \sim \frac{1}{4}, \frac{9}{4}, \frac{5}{4} - \frac{1}{2}\sigma^2$ .

•  $k \approx O(\sigma)$

In this region there are sub-regions where we have positive or negative values of  $\Delta$ . For example if we consider  $k = \sigma$  then  $\Delta = \frac{1}{4} - \frac{7}{4}\sigma^2 + \frac{215}{48}\sigma^4 - 5\sigma^6 + \frac{145}{72}\sigma^8 - \frac{1}{72}\sigma^{10}$  where we have two roots at  $\sigma = 0.60$  and  $\sigma = 1.05$ , between these roots  $\Delta < 0$  and beyond them it is positive.

•  $k \gg \sigma$

At this limit the last term in  $\Delta$  is dominant and we have  $\Delta \cong -\frac{(\sigma^2-1)^3}{27\sigma^6}k^6$ , so we have a changing sign around the  $\sigma = 1$ . For  $\sigma < 1$  there is one real positive root and for  $\sigma > 1$  there are three negative real roots. All roots are near the value of  $u = -k^2\frac{\sigma^2-1}{\sigma^2}$ .

For clarifying the behavior of roots for  $u$  we have presented some numerical values in the following table.

$\sigma$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{4}{3}$	$\frac{5}{3}$
$k = 0$	0.25, 2.25, 1.19	0.25, 2.25, 1.03	0.25, 2.25, 0.36	0.25, -0.14, 2.25
$k = 1$	9.97	2.00	2.26	2.30
$k = 2$	33.9	5.90	1.67	1.39
$k = 3$	73.9	12.1	0.23	-0.83
$k = 10$	802	126	-34.6	-52.8

Table 2: Real roots in different region of parameter space.

These numerical values show that for  $0 \leq \sigma \leq 1$  we always have at least one positive root where by increasing  $k$  only one positive root remains. But for  $1 < \sigma < 2$  by increasing  $k$  the number of positive roots reduces and for large enough values of  $k$  there is no positive root in this region.

## 6 Extended new massive gravity

In this section we consider higher curvature corrections to NMG and try to find their effects on spectrum of massless and massive perturbative solutions for both previous background metrics. The Lagrangian of extended NMG up to third order curvature terms [10]- [12] is given by

$$\mathcal{L}^{ENMG} = \sqrt{-g} \left( \kappa_3 R^\mu{}_\nu R^{\nu\rho} R_{\rho\mu} + \kappa_4 R R_{\mu\nu} R^{\mu\nu} + \kappa_5 R^3 \right), \quad (6.1)$$

where  $\kappa_3 = -\frac{2}{3m^4}$ ,  $\kappa_4 = \frac{3}{4m^4}$  and  $\kappa_5 = -\frac{17}{96m^4}$ . The equations of motion are obtained by the following energy-momentum tensor

$$\begin{aligned} T_{\mu\nu}^{ENMG} = & \kappa_3 \left( 3R_{\mu\alpha} R^{\alpha\beta} R_{\beta\nu} - \frac{1}{2} g_{\mu\nu} R^\alpha{}_\beta R^{\beta\rho} R_{\rho\alpha} + \frac{3}{2} [g_{\mu\nu} \nabla_\alpha \nabla_\beta (R^{\alpha\rho} R_\rho{}^\beta) + \square (R_\mu{}^\alpha R_{\alpha\nu}) - 2\nabla_\alpha \nabla_{(\mu} (R_{\nu)}{}^\beta R_{\beta}{}^\alpha)] \right) \\ & + \kappa_4 \left( R_{\mu\nu} R_{\alpha\beta} R^{\alpha\beta} + 2R R_\mu{}^\alpha R_{\alpha\nu} - \frac{1}{2} g_{\mu\nu} R R_{\alpha\beta} R^{\alpha\beta} + \square (R R_{\mu\nu}) + g_{\mu\nu} \nabla_\alpha \nabla_\beta (R^{\alpha\beta} R) - 2\nabla_\alpha \nabla_{(\mu} (R_{\nu)}{}^\alpha R) \right. \\ & \left. - [\nabla_\mu \nabla_\nu - g_{\mu\nu} \square] (R_{\alpha\beta} R^{\alpha\beta}) \right) + \kappa_5 \left( 3R_{\mu\nu} R^2 + 3[g_{\mu\nu} \square - \nabla_\mu \nabla_\nu] R^2 - \frac{1}{2} g_{\mu\nu} R^3 \right). \end{aligned} \quad (6.2)$$

In what follows we will study and solve the linearized form of (6.2) around the  $AdS_3$  and warped vacuum. Since the behavior of the extended NMG in different gauges is exactly similar to NMG we present only the results in the harmonic gauge.

## 6.1 $AdS_3$ vacuum

Similar to NMG we can find different properties of  $AdS_3$  vacuum solution in Extended NMG. In this case the equations of motion restrict the cosmological constant to

$$\Lambda = -\frac{8m^4l^4 + 2m^2l^2 + 1}{8m^4l^6}. \quad (6.3)$$

Once again if we perturb around the  $AdS_3$  vacuum we will find a similar fourth order differential equation (3.5) but in this case

$$\tilde{\mu} = \frac{1}{2l} \sqrt{\frac{8m^4l^4 + 4m^2l^2 + 3}{2m^2l^2 + 1}}. \quad (6.4)$$

The differential equation describes again a massless graviton mode and a massive graviton with mass square

$$\mathcal{M}^2 = \frac{8m^4l^4 - 4m^2l^2 - 1}{4l^2(2m^2l^2 + 1)}, \quad (6.5)$$

so the Tachyon free condition occurs for  $m^2l^2 \geq \frac{1+\sqrt{3}}{4}$  or  $-\frac{1}{2} \leq m^2l^2 \leq \frac{1-\sqrt{3}}{4}$ .

The highest weight solutions can be found by inserting the ansatz (3.10) into the highest weight equations (3.9) and determining  $h_{\mu\nu}$ . Substituting these values into the linearized equations of motion accompanied with the gauge conditions gives the following values for frequencies

$$\omega = k, k-2, k+4, k-1 \pm \frac{1}{2} \sqrt{\frac{8m^4l^4 + 4m^2l^2 + 3}{2m^2l^2 + 1}}, k+3 \pm \frac{1}{2} \sqrt{\frac{8m^4l^4 + 4m^2l^2 + 3}{2m^2l^2 + 1}}, \quad (6.6)$$

where the highest weight conditions restrict us to  $k=0$ . Comparing these results with (3.15) shows that only the mass of the massive modes gets correction.

We can confirm the above result by looking to the asymptotic behavior of the propagating solutions by inserting the ansatz (3.16) into the linearized equations of motion and using the gauge conditions. As an example we obtain the following differential equation for  $f_2(r)$

$$\begin{aligned} 0 = & (m^2l^2 + \frac{1}{2})r^8f_2^{(4)} + 18(m^2l^2 + \frac{1}{2})r^7f_2^{(3)} - ((m^4l^4 - \frac{191}{2}m^2l^2 - \frac{381}{8})r^2 + 2(m^2l^2 + \frac{1}{2})(k^2 - \omega^2))r^4f_2'' \\ & - ((7m^4l^4 - \frac{329}{2}m^2l^2 - \frac{651}{8})r^2 + 10(m^2l^2 + \frac{1}{2})(k^2 - \omega^2))r^3f_2' + ((-8m^4l^4 + 68m^2l^2 + 33)r^4 \\ & + (m^4l^4 - \frac{17}{2}m^2l^2 - \frac{33}{8})(k^2 - \omega^2)r^2 + (m^2l^2 + \frac{1}{2})(k^2 - \omega^2)^2)f_2. \end{aligned} \quad (6.7)$$

The series solution around the boundary behaves as  $r^{-B}$  such that  $B$  has the values

$$B = 2, 4, 3 \pm \frac{1}{2} \sqrt{\frac{8m^4l^4 + 4m^2l^2 + 3}{2m^2l^2 + 1}}. \quad (6.8)$$

Again the relation between  $\omega$  in the highest weight mode and  $B$  in the propagating mode is very similar to the NMG case (3.20) and only the square-root term has been corrected.

## 6.2 Warped- $AdS_3$ vacuum

If we consider the NMG Lagrangian and its curvature corrections and insert the vacuum solution (4.1) into the equations of motion  $T_{\mu\nu}^{NMG} + T_{\mu\nu}^{ENMG} = 0$ , then we will find the following values for mass parameter and

cosmological constant ( $\Delta = \sqrt{171\sigma^4 + 264\sigma^2 - 80}$ )

$$m^2 = m_{\pm}^2 = -\frac{3}{4l^2} \frac{21\sigma^2 - 4 \pm \Delta}{\sigma^2 - 4},$$

$$\Lambda = \Lambda_{\pm} = \frac{-783\sigma^6 + 2340\sigma^4 + 1008\sigma^2 - 576 \pm (-63\sigma^4 + 216\sigma^2 - 48)\Delta}{(\sigma^2 - 4)(21\sigma^2 - 4 \pm \Delta)^2 l^2}. \quad (6.9)$$

By taking  $l^2 > 0$  the following behaviors for mass parameter and cosmological constant will be obtained. The reality of cosmological constant constraints the warp factor to begin from  $\sigma_c = \frac{2\sqrt{-627+342\sqrt{6}}}{57} \cong 0.509$ .

$\sigma$	$\sigma_c \leq \sigma < \sigma_L$	$\sigma_L < \sigma < \sigma_R$	$\sigma_R < \sigma < 2$
$m_+^2$	$> 0$	$> 0$	$> 0$
$\Lambda_+$	$> 0$	$< 0$	$> 0$

Table 3: Behavior of  $m_+$  and  $\Lambda_+$  for  $\sigma_c \leq \sigma < 2$ .

$\sigma_L$  and  $\sigma_R$  are two real (between 0 and 2) roots of  $\Lambda_+ = 0$ . Numerically they are equal to  $\sigma_L \cong 0.558$  and  $\sigma_R \cong 1.802$ .

$\sigma$	$\sigma_c \leq \sigma < \frac{2}{\sqrt{15}}$	$\frac{2}{\sqrt{15}} < \sigma < \frac{2}{\sqrt{3}}$	$\frac{2}{\sqrt{3}} < \sigma < 2$
$m_-^2$	$> 0$	$< 0$	$> 0$
$\Lambda_-$	$> 0$	$> 0$	$< 0$

Table 4: Behavior of  $m_-$  and  $\Lambda_-$  for  $\sigma_c \leq \sigma < 2$ .

### 6.2.1 Massless and massive modes

Similar to NMG we use the ansatz (4.7) and put it into the equations (4.6). We achieve the same results as (4.9) and (4.10). If we substitute these results into the linearized equations of motion then we would find non-trivial solutions when the determinant of coefficients is zero. Like the NMG in harmonic gauge, here we have two polynomials  $P'_1$  and  $P'_2$ . The first polynomial is  $P'_1 = P_1$ , in other words we find again the massless mode of (4.11) and it does not receive any correction. The other polynomial which represents the massive modes gets correction and it can be written as

$$P'_2 = E_0(k, \sigma) + E_1(k, \sigma)u + E_2(k, \sigma)u^2 + E_3(k, \sigma)u^3 = 0, \quad (6.10)$$

where  $\omega = u^{\frac{1}{2}} + \frac{1}{2}$  and the functions  $E_i$  are given in the appendix B.

If we try to find the asymptotic behavior of propagating solutions again we will find the above results exactly.

## 7 Summary and Conclusions

In this paper we have mainly discussed about the stability of  $AdS$  and warped  $AdS$  vacua as solutions of the new massive gravity in three dimensions. First we found the equations of motion for NMG and then linearized

them around arbitrary background  $\bar{g}_{\mu\nu}$ . We determined the value of cosmological constant for each kind of solutions in terms of other parameters in the theory such as  $m, l, \sigma$ .

We observed that in the harmonic gauge one could write the linearized equations of motion as a product of decoupled operators (3.5) for the  $AdS$  vacuum. We should note that although there have not been found such a decomposition for the warped space our calculations might be a hint for finding it.

We considered the behavior of metric perturbations in the asymptotic limit from two points of view. In the first view, fluctuations satisfied in the highest weight equation in addition to the linearized equations of motion. While in the later, these equations decomposed to some differential equations for each perturbation that could be solved by analytical methods in the asymptotic limit. We have used two different gauge conditions to write the linearized equations of motion.

In warped  $AdS_3$  vacuum the value of frequencies  $\omega$ , coming from the highest weight formalism, was related to the radial fall off power parameter of the general propagating modes at the boundary. In fact in most cases we have studied, the propagating modes of metric fluctuations were also belong to the representations of isometry group of the background.

We observed that the existence of a mode in a vacuum, depends on the gauge choice. For example in the case of warped vacuum in the harmonic gauge we obtained two polynomials,  $P_1$  and  $P_2$ , while in  $h_{\mu\varphi} = 0$  gauge there was just one polynomial  $P_2$ . So it seems that the behavior of propagating solutions or highest weight modes depends on choosing of the gauge conditions and some modes are forbidden when we change the gauge. This can be seen for  $AdS_3$  vacuum as well, since as we found in harmonic gauge there is a highest weight mode while it becomes pure gauge when we go to the  $h_{\mu\varphi} = 0$  gauge.

We also discussed comprehensively about the stability of these vacuum perturbations at the asymptotic limit and found the domains of validity for parameters in different gauges. We showed that  $P_1$  polynomial describes the massless modes of the theory. For the squashed warped space-time there is always one possible allowed frequency but for the stretched warped space-time the stability is limited to some regions of space of parameters. We also showed that  $P_2$  which describes the massive modes has always negative energy both in the squashed and in the stretched warped space-time.

We have considered the extension of NMG, which are constructed from curvature terms and are consistent with the  $AdS/CFT$  context. This extended Lagrangian did not change the main results but only corrected the values of mass for the massive propagating modes of  $AdS_3$  background.

In warped space-time we showed that again there are two polynomials. A polynomial  $P'_1$  in the extended theory which is exactly equal to  $P_1$  so that the massless modes do not correct by higher curvature terms. In fact this encouraged us to look for this result in the TMG model. Again we found two polynomials in the harmonic gauge which one of them was exactly  $P_1$  and the other one was the result of [15] for TMG in  $h_{\mu\varphi} = 0$  gauge.

## Acknowledgment

A. G. would like to thanks D. Anninos, M. Guica and A. E. Mosaffa for very useful discussions. D. M. would like to thanks H. Golchin for discussions. This work was supported by Ferdowsi University of Mashhad under the grant 2/23391 (02/08/1391).

## A $S_0, S_1, S_2$ functions

### • The real functions in $h_{\mu\varphi} = 0$ gauge

$$\begin{aligned} S_0 &= 42(\omega-1)\sigma^{10} + (15\omega^3 - 45\omega^2 + 3(5k^2+1)\omega + 48 + 6k^2)\sigma^8 - (2\omega^5 + 7\omega^4 + (4k^2-54)\omega^3 + (9k^2+68)\omega^2 \\ &\quad + (2k^4 - 44k^2 - 8)\omega + 50k^2 + 2k^4)\sigma^6 + (2\omega^6 - 10\omega^5 + (4k^2+16)\omega^4 - (8k^2+8)\omega^3 + (2k^4+17k^2)\omega^2 \\ &\quad + (2k^4 - 57k^2)\omega + 8k^4 + 56k^2)\sigma^4 - 2k^2(4+k^2-4\omega+\omega^2)(k^2-\omega+2\omega^2)\sigma^2 + 2k^4(4+k^2-4\omega+\omega^2), \\ S_1 &= 42(\omega-1)\sigma^{10} + (15\omega^3 + 27\omega + 15k^2\omega - 51\omega^2 + 24)\sigma^8 - (2\omega^5 - 10\omega^4 + (4k^2+31)\omega^3 - (25k^2+68)\omega^2 \\ &\quad + (2k^4 + 41k^2 + 60)\omega - 18k^2 - 15k^4)\sigma^6 + (2\omega^5 - (2k^2+10)\omega^4 + (16k^2+16\omega^3 - (8+4k^4+40k^2)\omega^2 \\ &\quad + (51k^2+14k^4)\omega - 36k^2 - 23k^4 - 2k^6)\sigma^4 + 2k^2(4+k^2-4\omega+\omega^2)((2k^2+\omega^2-2\omega)\sigma^2 - k^2), \\ S_2 &= 42(k^2 - \omega(\omega-2))\sigma^8 - (15\omega^4 - 45\omega^3 - 6\omega^2 + (57k^2+72)\omega - 36k^2 - 15k^4)\sigma^6 + (2\omega^6 - 8\omega^5 + (6+2k^2)\omega^4 \\ &\quad + (8-26k^2)\omega^3 + (45k^2-2k^4-8)\omega^2 + (40k^2-18k^4) - 2k^6 + 19k^4 - 60k^2)\sigma^4\omega + 4(\omega^5 - 5\omega^4 + (7+2k^2)\omega^3 \\ &\quad + (\frac{3}{2}-4k^2)\omega^2 + (\frac{7}{2}k^2-7+k^4)\omega + k^4 - 2k^2)k^2\sigma^2 - 4(\omega + \frac{1}{2})(4+k^2-4\omega+\omega^2)k^4, \end{aligned}$$

• The real functions in harmonic gauge

$$\begin{aligned}
S_0 = & (42k^4 - (42\omega^2 - 420\omega + 336)k^2 - 84\omega^4 + 84\omega - 294\omega^2 + 294\omega^3)\sigma^{14} + (78k^6 + (99\omega^2 + 417\omega - 420)k^4 \\
& - (36\omega^4 - 624\omega^3 + 630\omega^2 + 996\omega - 1056)k^2 - 57\omega^6 + 606\omega^2 - 117\omega^4 - 375\omega^3 + 207\omega^5 - 264\omega)\sigma^{12} - (10k^8 \\
& + (28\omega^2 - 200\omega + 296)k^6 - (24\omega^4 - 398\omega^3 + 542\omega^2 + 1142\omega - 1034)k^4 - (4\omega^6 - 196\omega^5 + 266\omega^4 + 900\omega^3 \\
& - 1368\omega^2 - 548\omega + 1104)k^2 + 2\omega^8 - 2\omega^7 - 20\omega^6 - 2\omega^5 + 160\omega^4 - 134\omega^3 - 196\omega^2 + 192\omega)\sigma^{10} + (4k^{10} + (16\omega^2 \\
& - 32\omega + 30)k^8 + (24\omega^4 - 100\omega^3 + 168\omega^2 - 462\omega + 398)k^6 + (16\omega^6 - 108\omega^5 + 270\omega^4 - 664\omega^3 + 497\omega^2 + 1021\omega \\
& - 912)k^4 + (4\omega^8 - 44\omega^7 + 156\omega^6 - 242\omega^5 + 56\omega^4 + 450\omega^3 - 606\omega^2 - 56\omega + 384)k^2 - 4\omega^2(\omega^2 - 1)^2(\omega - 2)^3)\sigma^8 \\
& - (16k^8 + (52\omega^2 - 112\omega + 46)k^6 + (60\omega^4 - 264\omega^3 + 348\omega^2 - 324\omega + 156)k^4 + (28\omega^6 - 192\omega^5 + 426\omega^4 - 318\omega^3 \\
& - 154\omega^2 + 408\omega - 192)k^2 + 4\omega(\omega^2 - 1)(\omega^3 - 6\omega^2 + 4\omega + 6)(\omega - 2)^2)k^2\sigma^6 + (24k^6 + (60\omega^2 - 144\omega + 58)k^4 \\
& + (48\omega^4 - 228\omega^3 + 304\omega^2 - 62\omega - 88)k^2 + 4(3\omega^4 - 9\omega^3 - 3\omega^2 + 11\omega + 4)(\omega - 2)^2)k^4\sigma^4 + 4(k^2 + (\omega - 2)^2)k^8 \\
& - 4(k^2 + (\omega - 2)^2)(4k^2 + 3\omega^2 - 4(\omega + 1))k^6\sigma^2, \\
S_1 = & (84k^4 - 168k^2 - 42\omega^4 + 84\omega^3 + (42k^2 + 42)\omega^2 + (210k^2 - 84)\omega)\sigma^{14} - (78\omega^6 - 333\omega^5 + (99k^2 + 522)\omega^4 \\
& - (876k^2 + 537)\omega^3 - (36k^4 - 1443k^2 - 534)\omega^2 - (543k^4 + 600k^2 + 264)\omega - 57k^6 + 828k^4 + 240k^2)\sigma^{12} \\
& + (10\omega^8 - 58\omega^7 + (28k^2 + 80)\omega^6 + (28k^2 + 110)\omega^5 + (24k^4 - 196k^2 - 292)\omega^4 + (230k^4 - 22k^2 - 22)\omega^3 \\
& + (4k^6 - 702k^4 + 270k^2 + 364)\omega^2 + (144k^6 - 376k^4 - 588k^2 - 192)\omega - 2k^8 - 426k^6 + 860k^4 + 792k^2)\sigma^{10} \\
& - (4\omega^{10} - 28\omega^9 + (16k^2 + 64)\omega^8 - (84k^2 + 24)\omega^7 + (24k^4 + 154k^2 - 108)\omega^6 - (84k^4 + 114k^2 - 132)\omega^5 \\
& + (16k^6 + 132k^4 - 14k^2 + 8)\omega^4 - (28k^6 - 144k^4 - 294k^2 + 80)\omega^3 + (4k^8 + 58k^6 - 518k^4 - 474k^2 + 32)\omega^2 \\
& + (234k^6 + 191k^4 - 72k^2)\omega + 16k^8 - 585k^6 - 124k^4 + 384k^2)\sigma^8 + (16\omega^8 - 100\omega^7 + 4(13k^2 + 48)\omega^6 \\
& - 4(57k^2 + 5)\omega^5 + (60k^4 + 290k^2 - 304)\omega^4 - (156k^4 - 6k^2 + 216)\omega^3 + (28k^6 + 136k^4 - 194k^2 + 96)\omega^2 \\
& - 4(7k^6 - 32k^4 - 38k^2 + 24)\omega + (4k^6 + 38k^4 - 232k^2 - 176)k^2)\sigma^6 - (24\omega^6 - 132\omega^5 + 4(15k^2 + 52)\omega^4 \\
& - 4(51k^2 - 5)\omega^3 + (48k^4 + 178k^2 - 256)\omega^2 - (72k^4 - 54k^2 + 80)\omega + 12k^6 + 36k^4 - 80k^2 + 64)k^4\sigma^4 \\
& + 4(3k^2 + 4\omega^2 - 3\omega - 4)(k^2 + 4 - 4\omega + \omega^2)k^6\sigma^2 - 4k^8(k^2 + 4 - 4\omega + \omega^2), \\
S_2 = & (252\omega^3 - 252\omega^2 + (252k^2 - 252)\omega - 504k^2)\sigma^{12} + (270\omega^5 - 330\omega^4 + (540k^2 - 702)\omega^3 + (426 - 1080k^2)\omega^2 \\
& + (744 + 270k^4 - 516k^2)\omega + 984k^2 - 750k^4)\sigma^{10} - (24\omega^7 - 262\omega^6 + (712 + 72k^2)\omega^5 + (270 + 382k^2)\omega^4 - (72k^4 \\
& + 1564k^2 - 868)\omega^3 - (22k^4 - 1434k^2 + 340)\omega^2 - (24k^6 + 852k^4 - 992k^2 + 480)\omega + 1652k^4 - 456k^2 - 142k^6)\sigma^8 \\
& + (8\omega^9 - 52\omega^8 + (108 + 32k^2)\omega^7 - (152k^2 + 40)\omega^6 + (48k^4 + 216k^2 - 116)\omega^5 - (144k^4 + 340k^2 - 96)\omega^4 + (32k^6 \\
& - 108k^4 + 892k^2 + 36)\omega^3 - (40k^6 - 40k^4 + 356k^2 + 40)\omega^2 + (8k^8 + 954k^4 - 668k^2)\omega + 4k^8 + 340k^6 - 1086k^4 \\
& - 24k^2)\sigma^6 - (24\omega^7 - 132\omega^6 + 8(9k^2 + 26)\omega^5 - 4(63k^2 + 1)\omega^4 + 8(9k^4 + 19k^2 - 22)\omega^3 - 2(54k^4 - 31k^2 - 12)\omega^2 \\
& + (24k^6 - 56k^4 + 156k^2 + 48)\omega + 12k^6 + 270k^4 - 216k^2)k^2\sigma^4 + (24\omega^5 - 108\omega^4 + 4(12k^2 + 29)\omega^3 - 4(24k^2 \\
& - 11)\omega^2 + 8(3k^4 - 2k^2 - 7)\omega + 12k^4 + 88k^2 - 24)k^4\sigma^2 - 4(2\omega + 1)(k^2 + 4 - 4\omega + \omega^2)k^6.
\end{aligned}$$

## B Functions of $P'_2$ polynomial

$$E_0 = \left[ - (430272\sigma^{12} + (232704k^2 - 84096)\sigma^{10} - (70608k^4 - 285672k^2 + 696405)\sigma^8 - (2048k^6 + 176160k^4 - 573856k^2 + 158790)\sigma^6 + (6144k^6 + 129488k^4 - 6392k^2 - 32544)\sigma^4 - (6144k^6 + 12928k^4 + 65216k^2 - 2880)\sigma^2 - 11008k^4 + 16128k^2 + 2048k^6)\Delta + 5614272\sigma^{14} + (3034368k^2 + 3341952)\sigma^{12} - (923472k^4 - 5876136k^2 + 13288653)\sigma^{10} + (30720k^6 + 1409376k^4 - 5291328k^2 - 1399494)\sigma^8 - (116736k^6 - 1164336k^4 + 7862728k^2 - 3083592)\sigma^6 + (165888k^6 - 3181376k^4 + 5537120k^2 - 381312)\sigma^4 - (104448k^6 - 1849600k^4 + 1422592k^2 + 18432)\sigma^2 - 129024k^2 - 318464k^4 + 24576k^6 \right] / 5614272,$$

$$E_1 = \left[ (-29088\sigma^{10} + (17652k^2 - 56073)\sigma^8 - (768k^4 + 15280k^2 + 5218)\sigma^6 + (2048k^4 - 8484k^2 + 8000)\sigma^4 + (7584k^2 - 1792k^4 - 1744)\sigma^2 - 1472k^2 + 512k^4)\Delta + 379296\sigma^{12} - (230868k^2 - 1034721)\sigma^{10} + (11520k^4 - 3072k^2 + 410238)\sigma^8 + (478852k^2 - 255560 - 35840k^4)\sigma^6 + (43264k^4 + 328944k^2 + 5488)\sigma^4 + (98112k^2 + 10816 - 25088k^4)\sigma^2 - 14080k^2 + 6144k^4 \right] / 701784,$$

$$E_2 = \left[ (4413\sigma^8 + (3370 - 384k^2)\sigma^6 + (896k^2 - 1952)\sigma^4 + (-640k^2 + 640)\sigma^2 + 128k^2)\Delta - 57717\sigma^{10} - (89622 - 5760k^2)\sigma^8 - (13952k^2 - 27272)\sigma^6 + (1472 + 12160k^2)\sigma^4 + (-3328 - 5504k^2)\sigma^2 + 1536k^2 \right] / 350892,$$

$$E_3 = 32(15\sigma^2 + 4 - \Delta)\sigma^2(\sigma^2 - 1)^2 / 87723,$$

where  $\Delta = \sqrt{171\sigma^4 + 264\sigma^2 - 80}$ .

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